Simultaneous Nearest Neighbor Search

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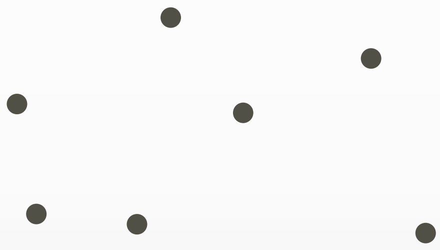
Cornell

Nearest Neighbor

- Dataset of n points P in a metric space (X, d_X)



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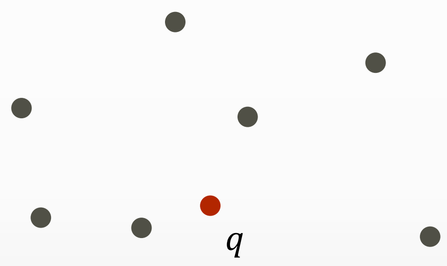

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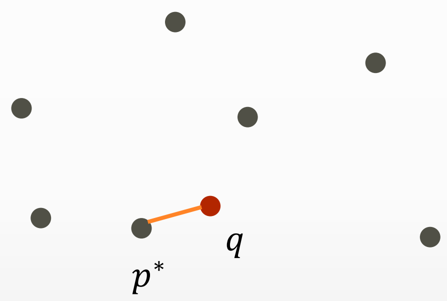

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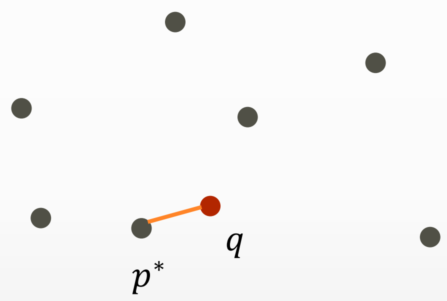

Approximate Nearest Neighbor

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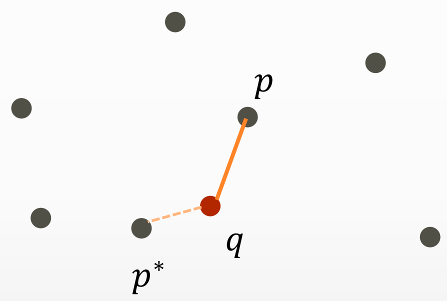

- Goal:
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What if

We have multiple queries

We need the results of the queries to be related.

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Example:

- Noisy image
- For each pixel find the true color
- Neighboring pixels have similar color

Simultaneous NN Problem

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(Felzenszwalb'15)

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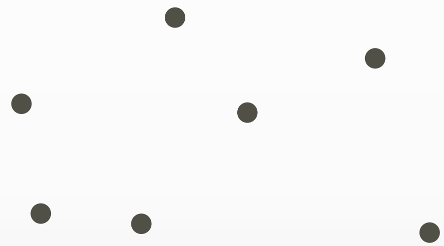

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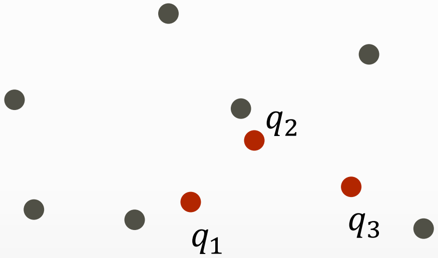

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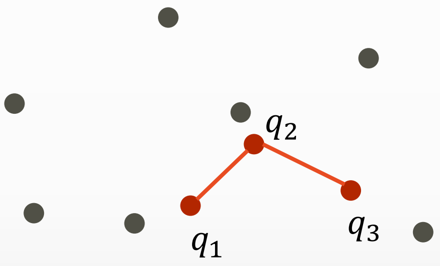

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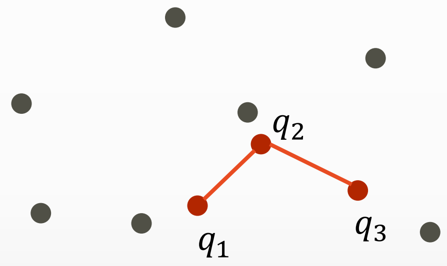

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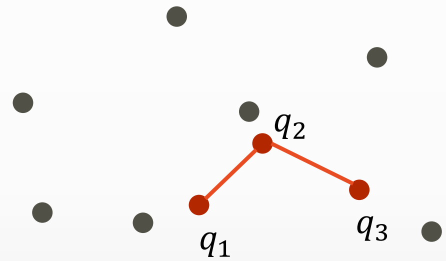

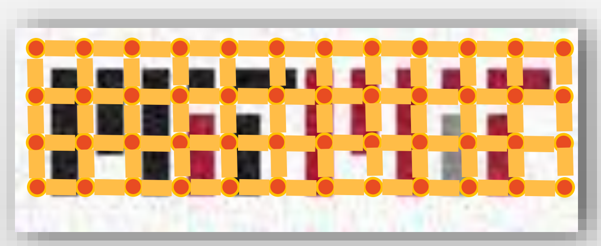

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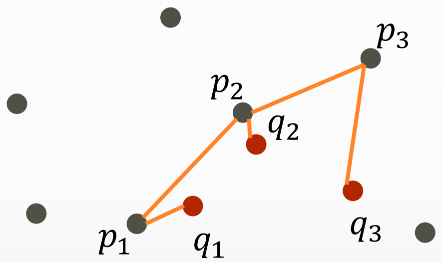

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$$\sum_{i=1}^{k} d_X(q_i, p_i) + \sum_{(q_i, q_j) \in E_G} d_X(p_i, p_j)$$

The Generalized SNN

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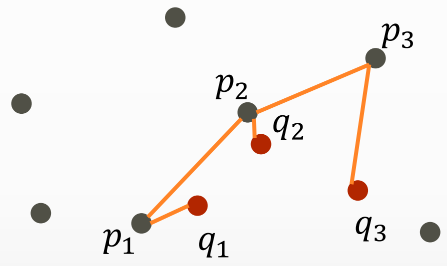

- Goal is to report $(p_1, \ldots, p_k)$, $p_i \in P$, that minimizes

$$\sum_{i=1}^{k} \kappa_i d_Y(q_i, p_i) + \sum_{(q_i, q_j) \in E_G} \lambda_{i,j} d_X(p_i, p_j)$$

Independent NN Algorithm

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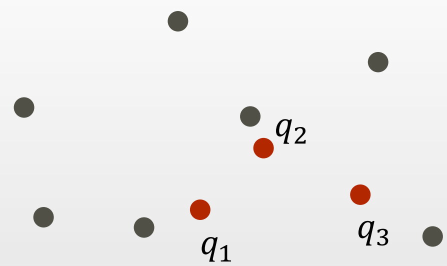

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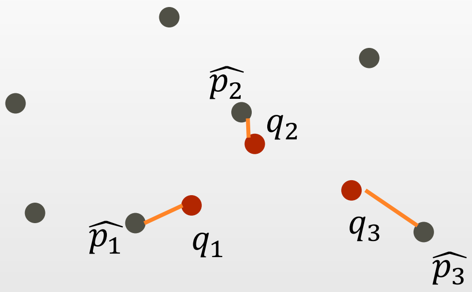

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- For each query point $q_i \in Q$
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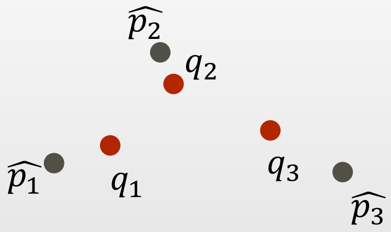

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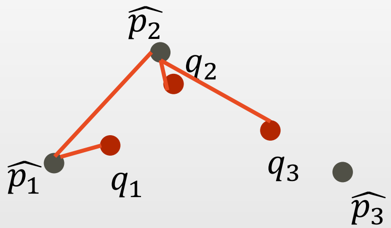

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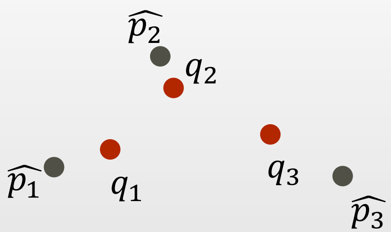

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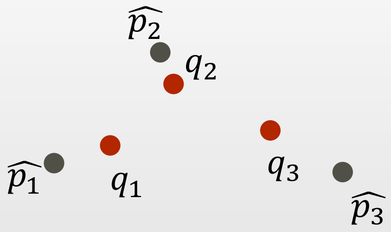

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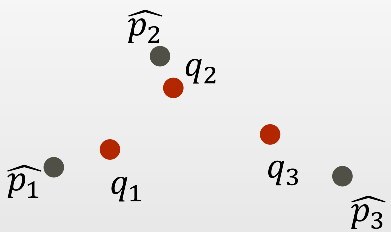

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➢ Any metric labeling β-approximate algorithm can be used on the reduced set , giving us $(\alpha \cdot \beta)$-approximate algorithm.

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Overview of the proof for

$$\alpha = O\left(\frac{\log k}{\log\log k}\right)$$

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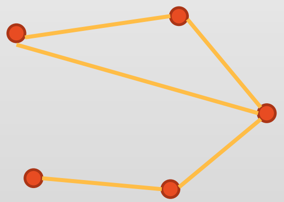

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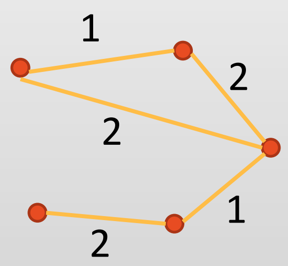

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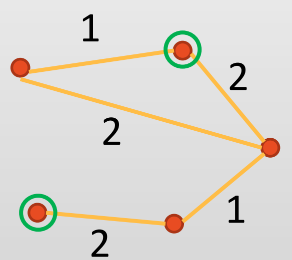

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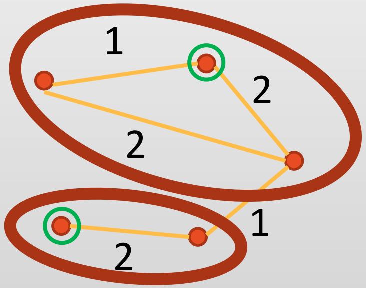
  - a set of terminals $T \subset V$
- The goal: find a mapping $f: V \to T$ s.t.
  - Each terminal is mapped to itself
  - Minimize $\sum_{(u,v)\in E} w(u,v) d(f(u), f(v))$



Cost = $1 \cdot d(t_1, t_2)$

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- $\Omega(\sqrt{\log |T|})$ integrality gap [FHRT03]
- Efficient if the number of terminals is low

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   - Using subtle properties of existing algorithms for 0-extension
   - Leads to an $\boldsymbol{O}(\frac{\boldsymbol{log\ k}}{\boldsymbol{log\ log\ k}})$ **approximation**

Experiments

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- **De-noising problem**
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- **Result:** empirical pruning gap α is very close to 1, at most 1.024

Experimental Results

Image

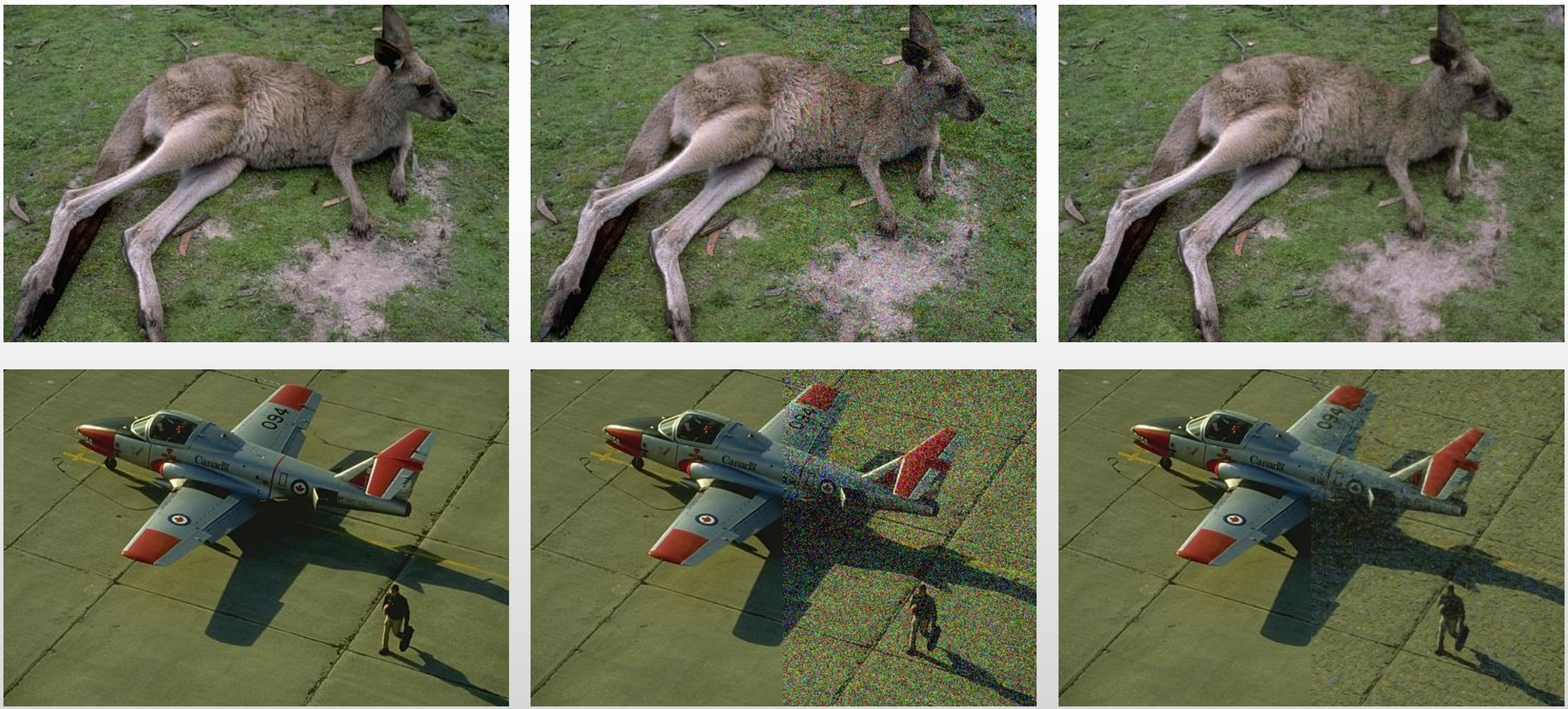
Noisy Image

De-noised using all colors

De-noised using noisy image colors

Experimental Results

Image | Half-Noisy | De-noised



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Thank You!